# Applicazioni del Principio di indifferenza Materiale

Nel caso della velocità e del suo gradiente, letti da un altro osservatore, si ha; poiché $x = R x'$,

$$\underline{v}' = \frac{d x'}{dt} = \dot{\underline{\underline{R}}}^T \underline{x} + \underline{\underline{R}}^T \dot{\underline{x}} = \dot{\underline{\underline{R}}}^T \underline{\underline{R}}\, \underline{x}' + \underline{\underline{R}}^T \underline{v} \; . \qquad (1)$$

Dunque $\underline{v}$ non si trasforma semplicemente in $R^T \underline{v}$, ma è influenzato anche dalla scelta del tensore antisimmetrico $\dot{R}^T R$; quindi $\underline{v}$ <u>non</u> può intervenire direttamente in un'equazione costitutiva. Per il suo gradiente si ha:

$$\nabla_{x'} \underline{v}' = \nabla_{x'} \left( \dot{\underline{\underline{R}}}^T \underline{\underline{R}}\, \underline{x}' + \underline{\underline{R}}^T \underline{v} \right) = \dot{\underline{\underline{R}}}^T \underline{\underline{R}}\, \nabla_{x'} \underline{x}' + \underline{\underline{R}}^T \nabla_{x'} \underline{v} =$$

$$= \dot{\underline{\underline{R}}}^T \underline{\underline{R}}\, \underline{\underline{I}} + \underline{\underline{R}}^T \nabla_x \underline{v} \frac{\partial \underline{x}}{\partial \underline{x}'} = \dot{\underline{\underline{R}}}^T \underline{\underline{R}} + \underline{\underline{R}}^T \nabla_x \underline{v}\, \underline{\underline{R}} = \qquad (2)$$

$$\dot{\underline{\underline{R}}}^T \underline{\underline{R}} + \underline{\underline{R}}^T \underline{\underline{W}}\, \underline{\underline{R}} + \underline{\underline{R}}^T \underline{\underline{D}}\, \underline{\underline{R}} \, ,$$

dove abbiamo usato la decomposizione del $\nabla v$ nella velocità di deformazione $D$ e nella vorticità $W$. Dunque:

$$\underline{\underline{D}}' = \underline{\underline{R}}^T \underline{\underline{D}}\, \underline{\underline{R}} \qquad \text{e} \qquad \underline{\underline{W}}' = \dot{\underline{\underline{R}}}^T \underline{\underline{R}} + \underline{\underline{R}}^T \underline{\underline{W}}\, \underline{\underline{R}} \, , \qquad (3)$$

poiché $\dot{\underline{\underline{R}}}^T \underline{\underline{R}} = -(\dot{R}^T R)^T$ per la relazione $R^T R = I$.

Quindi solo $D$ può comparire nelle equazioni costitutive.

$(4.32)_{(1)}$ si ha $\boxed{\underline{\underline{T}} = t\, \underline{\underline{F}}^c} = J\, \underline{\underline{t}}\, F^{-T}$, cioè in componenti

$$T_{iA} = J\, t_{ij} \frac{\partial X_A}{\partial x_j}, \quad S = \underline{\underline{F}}^{-1} \underline{\underline{t}} \quad \boxed{F^c_{ij} = (-1)^{i+j} \det F^{(ij)}}$$

$$\frac{\partial x}{\partial X} = \underline{\underline{F}} = \begin{pmatrix} e^t & 0 & (e^t-1) \\ 0 & 1 & (e^t-e^{-t}) \\ 0 & 0 & (1+3tX_3^2) \end{pmatrix} \Rightarrow J = (1+3tX_3^2)\, e^t > 0$$

$$F^c = \begin{pmatrix} (1+3tX_3^2) & -0 & 0 \\ 0 & -0 & e^t(1+3tX_3^2) \; -0 \\ (1-e^t) & -(e^{2t}-1) & e^t \end{pmatrix} \Rightarrow$$

$$\Rightarrow T = t\, F^c = \begin{pmatrix} (1+3tX_3^2)\, 3x_2 e^{-t} + x_1(1-e^t) & e^t(1+3tX_3^2)\, x_3 + x_1(e^{2t}-1) & (x_1 e^t) \\ (1+3tX_3^2)\, x_3 t + x_2(1-e^t) & 5x_3 e^t(1+3tX_3^2) + x_2(e^{2t}-1) & (e^t x_2) \\ x_1(1+3tX_3^2) + 7x_1(1-e^t) & x_2 e^t(1+7tX_3^2) + 7x_1(e^{2t}-1) & 7x_1 e^t \end{pmatrix}$$

$$= \begin{cases} T_{11} = (1+3tX_3^2)\, 3\left[X_2 e^{-t} + X_3(1-e^{-2t})\right] + (1-e^t)\left[X_1 e^t + X_3(e^t-1)\right] \\ T_{12} = (1-e^{2t})\left[X_1 e^t + X_3(e^t-1)\right] + (1+3tX_3^2)\, e^t X_3(1+tX_3^2) \\ \text{(e così via ritrovando le (4.57) del testo dell'esercizio).} \\ T_{13} = X_1 e^{2t} + X_3 e^t(e^t-1) \;;\; T_{23} = e^t X_2 + X_3(e^{2t}-1) \\ T_{21} = (1+3tX_3^2)(X_3+tX_3^3) + (1-e^t)\left[X_2 + X_3(e^t-e^{-t})\right] \\ T_{22} = (1-e^{2t})\left[X_2 + X_3(e^t-e^{-t})\right] + 5(1+3tX_3^2)(X_3+tX_3^3) \end{cases}$$

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$$F^{-1} = J^{-1} F^{cT} = (1+3tX_3^2)^{-1} e^{-t} F^{cT} = \begin{pmatrix} e^{-t} & 0 & \frac{1-e^t}{(1+3tX_3^2)e^t} \\ 0 & 1 & \frac{-e^t+e^{-t}}{1+3tX_3^2} \\ 0 & 0 & (1+3tX_3^2)^{-1} \end{pmatrix}$$

$$\underline{X} = \underline{\underline{F}}^{-1} \underline{x}, \quad \begin{cases} X_1 = (F^{-1})_{1i} \, x_i \\ X_2 = (F^{-1})_{2i} \, x_i \\ X_3 = (F^{-1})_{3i} \, x_i \end{cases}$$

$$S_{11} = (F^{-1} T)_{11} = e^{-t} T_{11} + (1-e^t) J^{-1} T_{31}$$

$$S_{12} = e^{-t} T_{12} + (1-e^t) J^{-1} T_{32} = S_{21}$$

$$S_{13} = e^{-t} T_{13} + (1-e^t) J^{-1} T_{33} = S_{31}$$

$$S_{22} = T_{22} + (1-e^{2t}) J^{-1} T_{32}$$

$$S_{23} = T_{23} + (1-e^{2t}) J^{-1} T_{33} = S_{32}$$

$$S_{33} = (1+3tX_3^2)^{-1} T_{33}$$

(ii) $$\rho_* \underline{b} = \rho_* \underline{a} - \underline{\mathrm{Div}} \, \underline{\underline{T}} \Rightarrow \underline{b} = \underline{a} - \frac{1}{\rho_*} \underline{\mathrm{Div}} \, \underline{\underline{T}}$$

$$\underline{v} = \begin{pmatrix} X_1 e^t + X_3 e^t \\ X_3 (e^t + e^{-t}) \\ X_3^3 \end{pmatrix} = \begin{pmatrix} (X_1 + X_3) e^t \\ X_3 (e^t + e^{-t}) \\ X_3^3 \end{pmatrix}, \quad \underline{a} = \begin{pmatrix} (X_1 + X_3) e^t \\ X_3 (e^t - e^{-t}) \\ 0 \end{pmatrix}$$

$$
\begin{cases}
T_{31} = \left[\left(1+3+X_3^2\right) + 7\left(1-e^t\right)\right]\left[X_1 e^t + X_3\left(e^t-1\right)\right] \\
T_{32} = 7\left(1-e^{2t}\right)\left[X_1 e^t + X_3\left(e^t-1\right)\right] + \left(1+3+X_3^2\right)\left[e^t X_2 + X_1\left(e^{2t}-1\right)\right] \\
T_{33} = 7e^t\left[X_1 e^t + X_3\left(e^t-1\right)\right]
\end{cases}
$$

Quindi $\left(\mathrm{Div}\,T\right)_1 = \dfrac{\partial T_{1i}}{\partial X_i} = e^t\left(1-e^t\right) + 0 + e^t\left(e^t-1\right) = \underline{0}$

$$\left(\mathrm{Div}\,T\right)_2 = \frac{\partial T_{2i}}{\partial X_i} = \frac{\partial T_{21}}{\partial X_1} + \frac{\partial T_{22}}{\partial X_2} + \frac{\partial T_{23}}{\partial X_3} = 0 + \left(1-e^{2t}\right) + \left(e^{2t}-1\right) = \underline{0}$$

$$\left(\mathrm{Div}\,T\right)_3 = \frac{\partial T_{3i}}{\partial X_i} = e^t\left[\left(1+3+X_3^2\right) + 7\left(1-e^t\right)\right] + e^t\left(1+3+X_3^2\right) +$$

$$+ 7e^t\left(e^t-1\right) = e^t\left[2\left(1+3+X_3^2\right) + \cancel{7\left(1-e^t\right)} + \cancel{7\left(e^t-1\right)}\right] = 2e^t\left(1+3+X_3^2\right).$$

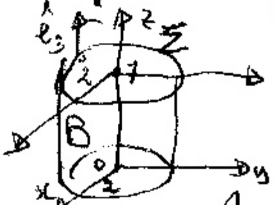
e si ha
$$
\begin{cases}
b_1 = d_1 - \dfrac{1}{\rho_*}\left(\mathrm{Div}\,T\right)_1 = \left(X_1 + X_3\right)e^t \\
b_2 = d_2 - \rho_*^{-1}\left(\mathrm{Div}\,T\right)_2 = X_3\left(e^t - e^{-t}\right) \\
b_3 = d_3 - \rho_*^{-1}\left(\mathrm{Div}\,T\right)_3 = -\dfrac{2}{\rho_*}\, e^t\left(1+3+X_3^2\right)
\end{cases}
$$

(4.35) (i) La superficie esterna è $\varphi = \frac{x_1^2}{4} + \frac{x_2^2}{4} - 1 = 0$

$$\Rightarrow \nabla\varphi = \left(\frac{x_1}{2}, \frac{x_2}{2}, 0\right) \quad e \quad |\nabla\varphi| = \sqrt{\frac{x_1^2}{4} + \frac{x_1^2}{4}} = \sqrt{1} = 1$$

dunque $\hat{n}$ è la normale esterna alla superficie laterale

Il flusso di calore è $q \cdot \hat{n} = \frac{1}{2}\left[x_1 (2x_1 t + x_3) + 3x_1 x_2^2 e^{-t}\right]$.




(ii) Il versore è $\hat{e}_3$ e quindi $q \cdot \hat{e}_3 = 5(x_1^2 + x_2^2) t e^{-tx_3}$;

valutandolo in $x_3 = 7$ si ha $q \cdot \hat{e}_3 = 5(x_1^2 + x_2^2) t e^{-7t}$ e

quindi il flusso totale è $Q = \int_{\Sigma} q \cdot \hat{e}_3 \, da = \int_0^2 dr \int_0^{2\pi} r \, d\theta \, q \cdot \hat{e}_3 =$

$$= 5te^{-7t} \cdot 2\pi \int_0^2 r^3 \, dr = 5te^{-7t} \, 2\pi \, \frac{2^4}{4} = 40\pi t e^{-7t}.$$

(4.36) (i) La deformazione è la stessa dell'esempio (4.32) e quindi

$$\underline{Q} = \underline{\underline{F}}^{cT} \underline{q} = \begin{pmatrix} (1+3tX_3^2) & 0 & (1-e^t) \\ 0 & J & (1-e^{2t}) \\ 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} 2x_1 e^{-t} \\ 3x_2 e^{-t} \\ 5x_3 e^{-t} \end{pmatrix} =$$

$$= \begin{pmatrix} 2x_1 e^{-t} (1+3tX_3^2) + 5x_3 e^{-t}(1-e^t) \\ J 3x_2 e^{t} + (1-e^{2t}) 5x_3 e^{-t} \\ 5x_3 \end{pmatrix} = \begin{bmatrix} 2(1+3tX_3^2)\left[X_1 + X_3(1-e^{-t})\right] + 5(e^{-t}-1)(tX_3^3 + X_3) \\ 3(1+3tX_3^2)\left[X_2 + X_3(e^t - e^{-t})\right] + 5(e^{-t}+e^{t})(tX_3^3 + X_3) \\ 5(tX_3^3 + X_3) \end{bmatrix}$$

(ii) Si ha $\phi = x_1 + x_2 \Rightarrow \nabla\phi = (1,1,0) \Rightarrow$

$\Rightarrow \hat{N} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$. Valutiamo $\underline{q}(\underset{x_1}{1}, \underset{x_2}{-e^{-t}}, \underset{x_3}{0}) =$

$$= \begin{pmatrix} 2e^{-t} \\ -3e^{-2t} \\ 0 \end{pmatrix} \Rightarrow Q = \begin{pmatrix} 2e^{-t} \\ -3e^{-t} \\ 0 \end{pmatrix} \Rightarrow \boxed{Q \cdot \hat{N} = -\frac{\sqrt{2}}{2} e^{-t}}$$

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$$\theta\, dS = \theta \frac{\partial S}{\partial F} dF + \theta \frac{\partial S}{\partial \theta} d\theta = \frac{\partial e}{\partial F} dF - \frac{T}{\rho^\circ} dF + \frac{\partial e}{\partial \theta} d\theta$$

$$= \left(\frac{\partial e}{\partial F} dF + \frac{\partial e}{\partial \theta} d\theta\right) - \frac{T}{\rho^\circ} dF = de - \frac{T}{\rho^\circ} dF$$

Helmholtz

$$d\psi = de - \theta\, dS - S\, d\theta = -S\, d\theta + \rho_0^{-1} \underline{\underline{S}} \cdot d\underline{\underline{E}}$$

$$\shortparallel$$

$$d\psi = \frac{\partial \psi}{\partial \theta} d\theta + \frac{\partial \psi}{\partial \underline{\underline{E}}} \cdot d\underline{\underline{E}}$$

$$\underline{\underline{T}} = \underline{\underline{t}}\, \underline{\underline{F}}^{-T} \cong \underline{\underline{t}}\left[\left(1+\operatorname{Div} u\right)I - \left(\operatorname{Grad} u\right)^T\right] =$$

$$= \underline{\underline{t}} + \operatorname{Div} \underline{u}\, \underline{\underline{t}} - \underline{\underline{t}}\left(\operatorname{Grad} u\right)^T \cong \underline{\underline{t}}$$

$$\underline{\underline{T}} = \underline{\underline{F}}\,\underline{\underline{S}} \qquad \underline{\underline{S}} = \underline{\underline{F}}^{-1}\underline{\underline{T}} = \left(I - \operatorname{Grad} u\right)\underline{\underline{T}} = \underline{\underline{T}} - \operatorname{Grad} u\, \underline{\underline{T}} =$$

$$\cong \underline{\underline{t}} - \left(\operatorname{Grad} u\right)\underline{\underline{t}} \cong \underline{\underline{t}}$$

$$\frac{\partial}{\partial X_A} = \frac{\partial}{\partial x_i}\frac{\partial x_i}{\partial X_A} = F_{iA}\frac{\partial}{\partial x_i} = \left(\delta_{iA} + \operatorname{Grad} u_{iA}\right)\frac{\partial}{\partial x_i} =$$

$$= \frac{\partial}{\partial x_A} + \left(\operatorname{Grad} u\right)_{iA}\frac{\partial}{\partial x_i}$$

$$\underline{\underline{E}} = \frac{1}{2}\left(\underline{\underline{C}} - I\right) = \frac{1}{2}\left(F^T F - I\right)$$

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$$\underline{\underline{F}}(0) = \underline{\underline{I}}, \quad \underline{\underline{E}}(0) = \frac{1}{2}\left(\underline{\underline{I}} - \underline{\underline{I}}\right) = \underline{\underline{0}}$$

$$e\left(J_1, J_2, J_3\right) = e\left(0,0,0\right) +$$

$$+ \left.\frac{\partial e}{\partial J_1}\right|_0 J_1 + \left.\frac{\partial e}{\partial J_2}\right|_0 J_2 + \left.\frac{\partial e}{\partial J_3}\right|_0 J_3 + \frac{1}{2}\left.\frac{\partial^2 e}{\partial J_i J_k}\right|_0 J_i J_k + O\left(\varepsilon^3\right) =$$

$$= e(0,0,0) + \frac{\partial e}{\partial J_1}(0,0,0)\, J_1 + \frac{\partial e}{\partial J_2}(0,0,0)\, J_2 + \frac{1}{2}\frac{\partial^2 e}{\partial J_1^2} J_1^2 + \tilde{O}\left(\varepsilon^3\right)$$

0 perché $\underline{\underline{t}}(0) = \underline{\underline{0}}$

$$\underline{\underline{S}} = \underline{\underline{t}} = \rho^* \frac{\partial e}{\partial \underline{\underline{E}}} = \rho^*\left[\left.\frac{\partial e}{\partial J_1}\right|_0 I + \left.\frac{\partial e}{\partial J_2}\right|_0 2E + \frac{1}{\cancel{2}}\left.\frac{\partial^2 e}{\partial J_1^2}\right|_0 \cancel{2} J_1 I + \tilde{O}\left(\varepsilon^2\right)\right]$$

$$\cong 2\left(\rho^* \left.\frac{\partial e}{\partial J_2}\right|_0\right) E + \rho^* \frac{\partial^2 e}{\partial J_1^2}\left(\operatorname{tr} E\right) I = \boxed{\lambda\left(\operatorname{tr} E\right)\underline{\underline{I}} + 2\mu \underline{\underline{E}} \cong \underline{\underline{t}}}$$

$$\Rightarrow e \simeq \frac{\nu}{2\rho} J_1^2 + \frac{\mu}{\rho} J_2 \quad , \text{con } \nu = \rho^* \frac{\partial e}{\partial J_1}\Big|_0 \, , \mu = \rho \frac{\partial e}{\partial J_2}$$

$$\lambda = \nu + \frac{2}{3}\mu \qquad 2\mu \mid$$

$$\frac{\partial t_{ij}}{\partial x_j} = \nu\, \delta_{ij} \frac{\partial^2 u_k}{\partial x_k \partial x_j} + \mu \left( \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right)$$

$$= \nu \frac{\partial^2 u_k}{\partial x_k \partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} + \mu \frac{\partial^2 u_k}{\partial x_k \partial x_i}$$

$$\operatorname{div} \underline{\underline{t}} = (\nu + \mu) \nabla (\operatorname{div} \underline{u}) + \mu \Delta \underline{u}$$

$$\frac{\partial \underline{u}}{\partial t} = \underline{d} \exp( \cdot ) (-i\nu) \, , \quad \frac{\partial^2 \underline{u}}{\partial t} = (-i\nu)^2 \underline{d} \exp(\cdot)$$

$$-\nu^2 \underline{u} \quad = -\nu^2 \underline{d} \exp( \cdot )$$

$$\frac{\partial \underline{u}_i}{\partial x_j} = d_i \exp( \cdot ) i \frac{\partial (x_k n_k)}{\partial x_j} = d_i \exp( \cdot ) i \delta_{kj} n_k =$$

$$= d_i \exp(\cdot)\, i n_j = i(\underline{d} \otimes \hat{n})_{ij} \exp( \cdot )$$

$$= i(\underline{u} \otimes \hat{n})_{ij}$$

$$\frac{\partial^2 \underline{u}}{\partial \underline{x}^2} = \nabla^2 \underline{u} = -(\underline{d} \otimes \hat{n} \otimes \hat{n}) \exp( \cdot ) = -\underline{u} \otimes \hat{n} \otimes \hat{n}$$

$$\Delta \underline{u} = -\underline{u} (\hat{n} \cdot \hat{n}) = -\underline{u} \quad , \quad \nabla \operatorname{div} \underline{u} = -(\underline{u} \cdot \hat{n}) \hat{n}$$

$$-\rho V^2 \underline{u} + (\nu+\mu)(\underline{u}\cdot\hat{n})\hat{n} + \mu\underline{u} = \underline{0}$$

$$(\rho V^2 - \mu)\underline{u} = (\nu+\mu)(\underline{u}\cdot\hat{n})\hat{n}$$

$$(\rho V^2 - \mu)\underline{d} = (\nu+\mu)(\underline{d}\cdot\hat{n})\hat{n}$$

$$V^2 = \frac{\mu}{\rho} \iff V = \pm\sqrt{\frac{\mu}{\rho}} \Rightarrow \mu > 0$$

Es.: Fare caso dell'acciaio con onda piana $x_1+x_2+x_3=0$.

Dim.: Si ha $\hat{n} = \frac{\sqrt{3}}{3}(1,1,1) \Rightarrow \underline{d}\cdot\hat{n} = 0 \iff (d_1+d_2+d_3)\frac{\sqrt{3}}{3} = 0$

$\Rightarrow \underline{d}^1 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$, e $\underline{d}^2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$. onde trasversali, mentre

$\underline{d}^3 = \alpha\hat{n} = \alpha\frac{\sqrt{3}}{3}(1,1,1)$ è longitudinale. Si ha, per le onde trasversali: $V_\pm^{1,2} = \pm\sqrt{\frac{\mu}{\rho}} = \pm\sqrt{\frac{77440}{7{,}86}} = \pm\sqrt{9852{,}42} \simeq \pm 99{,}26$

mentre $V_\pm^3 = \pm\sqrt{\frac{\nu+2\mu}{\rho}} = \pm\sqrt{\frac{\lambda+\frac{4}{3}\mu}{\rho}} = \pm\sqrt{\frac{150.300+\frac{4}{3}77.400}{7.86}} \simeq \pm\sqrt{32258{,}69} \simeq \pm 179{,}61$

con $\nu = \lambda - \frac{2}{3}\mu$, cioè $\nu+2\mu = \lambda - \frac{2}{3}\mu + 2\mu = \lambda + \frac{4}{3}\mu$

e $\lambda$ è il modulo di compressibilità

25'

densita.png

| Materiale | Densità a 20°C ($g/cm^3$) | | Materiale | Densità a 20°C e 1 atm (g/L) |
|---|---|---|---|---|
| acqua(a 4°C) | 1,00 | | aria | 1,29 |
| acciaio | 7,86 | | azoto | 1,25 |
| alcool etilico | 0,79 | | cloro | 3,0 |
| alluminio | 2,7 | | elio | 0,18 |
| legno | da 0,8 a 0,9 | | idrogeno | 0,089 |
| olio di oliva | 0,92 | | ossigeno | 1,43 |

Costanti di Lamè Mezzo Continuo.jpg

| Materiale | $\lambda$(MPa) | $\mu$(MPa) | $K$(MPa) |
|---|---|---|---|
| Acciaio | 150300 | 77440 | 202000 |
| Alluminio | 34300 | 26950 | 52270 |
| Argento | 22860 | 29100 | 42260 |
| Ferro | 82400 | 82400 | 137300 |
| Piombo | 11700 | 6000 | 15700 |
| Rame | 28300 | 46200 | 59150 |
| Stagno | 98800 | 16100 | 109500 |
| Titanio | 72860 | 44650 | 102630 |
| Tungsteno | 192900 | 151560 | 294000 |
| Zinco | 57100 | 38100 | 82500 |