$$\underline{t}_1 = \underline{\underline{T}}\,\hat{\underline{n}} = -p\,\underline{\underline{I}}\,\hat{\underline{n}} = -p\,\hat{n}$$

$$(-\operatorname{div} T)_i = -\frac{\partial t_{ij}}{\partial x_j} = +\frac{\partial p}{\partial x_j}\delta_{ij} = \frac{\partial p}{\partial x_i} = (\nabla p)_i$$

$$-t_{ij}v_j = +p\,\delta_{ij}v_j = p\,v_i\,,$$

$$\mathcal{P}^{(i)} = -\underline{\underline{T}}\cdot\underline{\underline{D}} = +p\,\delta_{ij}D_{ij} = +p\,D_{ii} = +p\,L_{ii} =$$

$$= +p\,\frac{\partial v_i}{\partial x_i} = +p\operatorname{div}\underline{v}\,, \qquad \operatorname{div} v = -\frac{1}{\rho}\dot{\rho} = -\frac{1}{\rho}\frac{d\rho}{dt}$$

$$\delta e_*^{(i)} = \frac{1}{\rho_*}\mathcal{P}_*^{(i)}dt = \frac{J}{\rho_*}\mathcal{P}^{(i)}dt = \frac{1}{\rho}p\operatorname{div} v\,dt =$$

$$= -\frac{1}{\rho^2}p\frac{d\rho}{\cancel{dt}}\cancel{dt} \qquad V = \frac{1}{\rho} = \frac{J}{\rho_*}$$

$$dV = -\frac{1}{\rho^2}d\rho$$

Fluidi reali

$$\underline{\underline{T}} = \nu\operatorname{div} v\,\underline{\underline{I}} + 2\mu D^D + \frac{2}{3}\mu\operatorname{div} v\,\underline{\underline{I}}$$

$$= \left(\nu + \frac{2}{3}\mu\right)(\operatorname{div} v)\,\underline{\underline{I}} + 2\mu D^D$$

$$\frac{\partial \rho}{\partial t} v_j + \rho \frac{\partial v_j}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} v_j + \rho v_i \frac{\partial v_j}{\partial x_i} + \frac{\partial p}{\partial x_i}\delta_{ij} - \frac{\partial \sigma_{ij}}{\partial x_i} =$$

$$= \cancel{-\frac{\partial(\rho v_i)}{\partial x_i}} v_j + \rho \frac{\partial v_j}{\partial t} + \cancel{\frac{\partial(\rho v_i)}{\partial x_i}} v_j + \rho v_i \frac{\partial v_j}{\partial x_i} + \frac{\partial p}{\partial x_j} - \frac{\partial \sigma_{ij}}{\partial x_i} =$$

$$= \rho\left(\frac{\partial v_j}{\partial t} + \frac{\partial v_j}{\partial x_i} v_i\right) + \frac{\partial p}{\partial x_j} - \frac{\partial \sigma_{ji}}{\partial x_i} =$$

$$= \left(\rho \frac{dv}{dt} + \nabla p - \operatorname{div} \sigma\right)_j$$

$$\frac{de}{dt} = \frac{\partial e}{\partial \rho}\dot{\rho} + \underline{\frac{\partial e}{\partial \theta}\dot{\theta}} = \underline{-\rho \frac{\partial e}{\partial \rho}\operatorname{div} v} + \frac{\partial e}{\partial \theta}\frac{d\theta}{dt}$$

$$\rho \frac{\partial e}{\partial \theta}\frac{d\theta}{dt} = \underline{\rho^2 \frac{\partial e}{\partial \rho}} \operatorname{div} v - p \operatorname{div} v + \sigma \cdot D - \operatorname{div} q + r$$

$$\boxed{\underline{\Theta \le -\frac{r}{\theta}} + \frac{1}{\theta}\operatorname{div} q - \frac{1}{\theta^2} \underline{q} \cdot \underline{\nabla}\theta + \rho \frac{\partial s}{\partial \theta}\frac{d\theta}{dt} - \rho^2 \frac{\partial s}{\partial \rho}\operatorname{div} v} \overset{(8.25)}{=}$$

$$\overset{(8.26)}{=} -\frac{1}{\theta^2}\underline{q}\cdot\nabla\theta - \frac{r}{\theta} + \frac{1}{\theta}\operatorname{div} q - \underline{\rho^2 \frac{\partial s}{\partial \rho}} \operatorname{div} \underline{v} + \cancel{\rho} \frac{\frac{\partial s}{\partial \theta}}{\cancel{\rho}\frac{\partial e}{\partial \theta}} \Big\{ \underline{r} +$$

$$+ \left(\rho^2 \frac{\partial e}{\partial \rho} - p\right) \underline{\operatorname{div}} \underline{v} + \underline{\sigma} \cdot \underline{\underline{D}} - \underline{\operatorname{div} q} \Big\} = -\theta^{-2} \underline{q} \cdot \underline{\nabla}\theta +$$

$$\underline{+\frac{1}{\theta}\left(\operatorname{div} \underline{q} - r\right)} - \frac{\frac{\partial s}{\partial \theta}}{\frac{\partial e}{\partial \theta}} \left(\underline{\operatorname{div} q - r}\right) + \operatorname{div} v \left\{-\rho^2 \frac{\partial s}{\partial \rho} + \left(\rho^2 \frac{\partial e}{\partial \rho} - p\right)\frac{\frac{\partial s}{\partial \theta}}{\frac{\partial e}{\partial \theta}}\right\} + \underline{\sigma} \cdot \underline{D} \, \frac{\partial s}{\partial \theta} \Big/ \frac{\partial e}{\partial \theta}.$$

$$\vartheta\, dS = \vartheta \frac{\partial S}{\partial \rho} d\rho + \vartheta \frac{\partial S}{\partial \vartheta} d\vartheta =$$

$$= \frac{\partial e}{\partial \rho} d\rho - \frac{p}{\rho^2} d\rho + \frac{\partial e}{\partial \vartheta} d\vartheta = de - \frac{p}{\rho^2} d\rho$$

$$\delta \ell_*^{(i)} =$$

$$\underline{\underline{t}} = -p \underline{\underline{I}} + \underline{\underline{\sigma}}$$

$$\underline{\underline{t}}_e = -p \underline{\underline{I}}$$

$$\mathcal{P}_e^{(i)} = - \underline{\underline{t}}_e \cdot \mathbb{D} = p\, \mathbb{I} \cdot \mathbb{D} = p \,\mathrm{div}\, v$$

$$J = 1 \Rightarrow \rho = \rho_0$$

$\mu$ costante per $\vartheta$ = costante

$$\begin{cases} \mathrm{div}\, \boldsymbol{v} = 0 \\ \rho_0 \dfrac{\partial v_j}{\partial t} + \cancel{\rho_0 \dfrac{\partial v_i}{\partial x_i} v_j} + \rho_0 v_i \dfrac{\partial v_j}{\partial x_i} + \dfrac{\partial p}{\partial x_j} \underbrace{- 2\mu \dfrac{\partial D^0_{ij}}{\partial x_i}}_{= -\mu \Delta v_j} = \rho b_j \end{cases}$$

$$\underline{f} = \underline{\underline{t}}\, \hat{n} = \Big( -(p + \underset{\substack{\| \\ 0}}{\cancel{\pi}}) \mathbb{I} + 2\mu \mathbb{D}^0 \Big) \hat{n} = -p\, \hat{n} + \mu \left( \nabla v + (\nabla v)^T \right) \hat{n}$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \tilde{\rho}}{\partial x_i} (v_0^i + \tilde{v}^i) + (\rho_0 + \tilde{\rho})\, \mathrm{div}(v_0 + \tilde{v}) = 0$$

$$(\rho_0 + \tilde{\rho}) \left[ \frac{\partial \tilde{v}}{\partial t} + (\nabla \tilde{v})(v_0 + \tilde{v}) \right] + \nabla \tilde{p} = \underline{0}$$

**Esempio:** Velocità del suono nell'aria con $\rho_0 = \rho_* = 1,29$, l'onda essendo il piano $x_1 + 2x_2 + 2x_3 = 1$; e nell'equilibrio si deve avere $\underline{v}_0 = (1,1,0)\ m/s$ e $S_0 = 6,8726\ kJ/kg$ costanti calcolate a $\vartheta_0 = 20\,°C$ ($\simeq 293,15\,°K$) e $1\,atm$ ($g/L$). (*)

**Dim:** Dunque $\nabla\varphi = \nabla(x_1 + 2x_2 + 2x_3 - 1) = \begin{pmatrix}1\\2\\2\end{pmatrix} \Rightarrow \hat{n} = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}$

e $v_{0n} = \underline{v}_0 \cdot \hat{n} = \frac{1}{3} + \frac{2}{3} = 1\ m/s$. $\left.\frac{\partial p}{\partial \rho}\right|_0 = \frac{7}{5}\rho_0^{2/5}$, $\left.\frac{\partial p}{\partial S}\right|_0 = c_v^{-1}\rho_0^{7/5}$

$\Rightarrow (p_\rho)_0 = 1,55$, $(p_S)_0 = 1,992$ e sostituendo nelle (2.49) si ha

$$\begin{cases} (1 - v_F)\,\delta\rho + 1,29\,\delta v_n = 0 \\ 1,29\,(1 - v_F)\,\delta\underline{v} + (1,55\,\delta\rho + 1,992\,\delta S)\,\hat{n} = 0 \\ (1 - v_F)\,\delta S = 0 \end{cases}$$

1) Onda materiale: $v_F = 1\ m/s$ radice tripla con $\delta v_n = 0$: e $\delta\underline{v}$ tangente al piano cioè $0 = \delta v_n = \frac{1}{3}(v_1 + 2v_2 + 2v_3) \Leftrightarrow$ $\delta\underline{v}^1 = \begin{pmatrix}2\\-1\\0\end{pmatrix}$, $\delta\underline{v}^2 = \begin{pmatrix}2\\0\\-1\end{pmatrix}$; $\delta S$ qualsiasi e $\delta\rho = -1,285\,\delta S$.

Onda sonora di velocità $c_0 = \sqrt{p_\rho|_0} \simeq 1,245\ m/s$, $v_F = 1 \pm 1,245 = \begin{cases}2,245\\-0,245\end{cases}$ $\delta\underline{v}^3 = \frac{2}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}$.

(*) Per un gas perfetto (biatomico come l'aria) si ha $p = \rho^\gamma e^{\frac{S-S_0}{c_v}}$ con $\gamma = \frac{7}{5}$ e $c_v = 0.717$ e dunque $\left.\frac{\partial p}{\partial \rho}\right|_0 = \gamma\rho^{\gamma-1}\left.e^{\frac{S-S_0}{c_v}}\right|_0 = \frac{7}{5}\rho_0^{2/5} = 1,4 \times 1,107 = 1,55$

e $\left.\frac{\partial p}{\partial S}\right|_0 = \rho^\gamma c_v^{-1}\left.e^{\frac{S-S_0}{c_v}}\right|_0 = \rho_0^{7/5} c_v^{-1} = 1,428 \times 1,395 = 1,992$

| Dati fondamentali - (T=300K, p=1bar) | | | | | | |
|---|---|---|---|---|---|---|
| | massa molecolare [kg/kmol] | $R_1$ [J/kgK] | $c_v$ [J/kgK] | $c_p$ [J/kgK] | Press. crit. [bar] | Temp. crit. [K] |
| Azoto | 28.01 | 296.8 | 743 | 1039 | 33.9 | 126 |
| Idrogeno | 2.018 | 4124 | 10183 | 14307 | 12.9 | 33.2 |
| CO | 28.01 | 296.8 | 744 | 1040 | 35 | 133 |
| Ossigeno | 32.00 | 259.8 | 658 | 918 | 50.5 | 154 |
| Aria | 28.97 | 287 | 718 | 1005 | 37.7 | 133 |
| $CO_2$ | 44.01 | 188.9 | 657 | 846 | 73.9 | 304 |
| Vapor acqueo ($p=p_{sat}$) | 18.02 | 461.8 | 1440 | 1900 | 220.9 | 647 |
| **Energia interna gas perfetti u [kJ/kg]** | | | | | | |
| T[K] | Azoto | Idrogeno | CO | Ossigeno | Aria | $CO_2$ |
| 260 | 193.0 | 2611.8 | 193.0 | 169.3 | 185.5 | 131.5 |
| 300 | 222.7 | 3012.5 | 222.7 | 195.5 | 214.5 | 156.9 |
| 400 | 296.9 | 4052.0 | 298.4 | 262.3 | 287.0 | 227.9 |
| 500 | 371.9 | 5101.5 | 373.5 | 332.0 | 361.0 | 307.1 |
| 600 | 446.8 | 6150.5 | 449.9 | 405.3 | 436.3 | 392.9 |
| 700 | 526.4 | 7200.0 | 530.9 | 480.2 | 513.1 | 484.3 |
| 800 | 608.9 | 8271.5 | 614.8 | 558.8 | 592.8 | 580.6 |
| 900 | 692.8 | 9343.5 | 698.6 | 638.8 | 675.3 | 680.7 |
| 1000 | 780.4 | 10433.0 | 787.2 | 721.4 | 759.4 | 783.5 |
| 1200 | 958.1 | 12659.0 | 968.5 | 890.7 | 936.1 | 997.9 |
| 1400 | 1142.5 | 14967.5 | 1157.3 | 1063.8 | 1115.8 | 1219.9 |
| 1600 | 1332.8 | 17361.0 | 1349.2 | 1240.7 | 1298.0 | 1447.7 |
| 1800 | 1526.2 | 19857.0 | 1544.1 | 1421.7 | 1489.4 | 1679.2 |
| 2000 | 1722.4 | 22418.5 | 1743.3 | 1604.0 | 1679.2 | 1914.1 |
| 2200 | 1920.3 | 25039.5 | 1945.7 | 1788.7 | 1873.3 | 2152.5 |
| 2400 | 2121.1 | 27725.5 | 2147.9 | 1977.7 | 2068.7 | 2392.0 |
| 2600 | 2323.3 | 30474.0 | 2350.3 | 2169.0 | 2265.9 | 2634.8 |
| 2800 | 2527.1 | 33287.5 | 2552.5 | 2363.0 | 2496.0 | 2877.7 |
| 3000 | 2732.4 | 36139.0 | 2761.0 | 2558.1 | 2668.6 | 3122.9 |
| **Entalpia gas perfetti h [kJ/kg]** | | | | | | |
| T[K] | Azoto | Idrogeno | CO | Ossigeno | Aria | $CO_2$ |
| 260 | 270.2 | 3684.0 | 270.2 | 236.9 | 260.1 | 180.6 |
| 300 | 311.8 | 4259.6 | 311.8 | 273.4 | 300.6 | 213.6 |
| 400 | 415.7 | 5714.8 | 417.1 | 366.2 | 401.8 | 303.5 |
| 500 | 520.4 | 7180.0 | 522.0 | 461.9 | 504.5 | 401.6 |
| 600 | 625.0 | 8644.7 | 628.0 | 561.2 | 608.5 | 506.2 |
| 700 | 734.3 | 10109.9 | 738.7 | 662.0 | 714.0 | 616.6 |
| 800 | 846.4 | 11597.1 | 852.3 | 766.7 | 822.4 | 731.8 |
| 900 | 960.0 | 13084.4 | 965.9 | 872.6 | 933.5 | 850.8 |
| 1000 | 1077.3 | 14590.0 | 1084.1 | 981.3 | 1046.4 | 972.5 |
| 1200 | 1314.5 | 17647.4 | 1324.8 | 1202.5 | 1280.5 | 1224.7 |
| 1400 | 1558.2 | 20787.3 | 1573.0 | 1427.5 | 1517.6 | 1484.4 |
| 1600 | 1807.8 | 24012.2 | 1824.3 | 1656.4 | 1757.1 | 1750.0 |
| 1800 | 2060.7 | 27339.6 | 2078.5 | 1889.4 | 2006.0 | 2019.3 |
| 2000 | 2316.3 | 30732.5 | 2337.1 | 2123.6 | 2253.2 | 2292.0 |
| 2200 | 2573.6 | 34184.9 | 2598.9 | 2360.3 | 2504.7 | 2568.2 |
| 2400 | 2833.7 | 37702.3 | 2860.5 | 2601.2 | 2757.5 | 2845.4 |
| 2600 | 3095.3 | 41282.2 | 3122.3 | 2844.5 | 3012.1 | 3126.1 |
| 2800 | 3358.5 | 44927.1 | 3383.9 | 3090.4 | 3299.6 | 3406.8 |
| 3000 | 3623.2 | 48610.0 | 3651.8 | 3337.5 | 3529.5 | 3689.8 |
| **Entropia gas perfetti alla pressione di 1 bar $s_r$ [kJ/kg K]** | | | | | | |
| T[K] | Azoto | Idrogeno | CO | Ossigeno | Aria | $CO_2$ |
| 260 | 6.7036 | 65.4000 | 6.3214 | 6.2906 | 6.7276 | 4.7477 |
| 300 | 6.8536 | 65.4500 | 7.0714 | 6.4219 | 6.8726 | 4.8659 |
| 400 | 7.1536 | 69.6500 | 7.3714 | 6.6906 | 7.1626 | 5.1227 |
| 500 | 7.3893 | 72.9000 | 7.6071 | 6.9031 | 7.3904 | 5.3409 |
| 600 | 7.5857 | 75.6500 | 7.7964 | 7.0844 | 7.5803 | 5.5318 |
| 700 | 7.7500 | 77.8500 | 7.9714 | 7.2313 | 7.7459 | 5.7045 |
| 800 | 7.8929 | 79.8500 | 8.1214 | 7.3781 | 7.8909 | 5.8591 |
| 900 | 8.0321 | 81.5500 | 8.2571 | 7.5031 | 8.0221 | 5.9977 |
| 1000 | 8.1536 | 83.1500 | 8.3821 | 7.6188 | 8.1395 | 6.1273 |
| 1200 | 8.3714 | 85.9500 | 8.6036 | 7.8188 | 8.3500 | 6.3568 |
| 1400 | 8.5571 | 88.3500 | 8.7929 | 7.9938 | 8.5330 | 6.5568 |
| 1600 | 8.7250 | 90.5500 | 8.9607 | 8.1469 | 8.6952 | 6.7341 |
| 1800 | 8.8714 | 92.5000 | 9.1107 | 8.2313 | 8.8402 | 6.8932 |
| 2000 | 9.0071 | 94.2500 | 9.2464 | 8.4063 | 8.9713 | 7.0364 |
| 2200 | 9.1321 | 95.9000 | 9.3714 | 8.5188 | 9.0922 | 7.1682 |
| 2400 | 9.2429 | 97.4500 | 9.4857 | 8.6219 | 9.2026 | 7.2886 |
| 2600 | 9.3664 | 98.9000 | 9.5893 | 8.7219 | 9.3027 | 7.4000 |
| 2800 | 9.4464 | 100.2500 | 9.6857 | 8.8125 | 9.3959 | 7.5045 |
| 3000 | 9.5393 | 101.5000 | 9.7786 | 8.8969 | 9.4891 | 7.6023 |

Dai valori dell'entropia di riferimento si deducono quelli per p qualunque aggiungendo $-R1 \log_n(p)$, p espressa in bar

$$s = s_r - R1 \log_n(p)$$

L' onda piana $x_1 + 2x_2 + 2x_3 = 1$ (31)

$\rho_0 = 1{,}29$ , $\underline{v}_0 = (1, 1, 0)$ m/s , $S_0 = 6{,}8726$

$\theta_0 = 20°C \simeq 293{,}16$ °K

$\varphi = x_1 + 2x_2 + 2x_3 - 1 \Rightarrow \nabla\varphi = (1, 2, 2)$ , $|\nabla\varphi| = \sqrt{1+4+4} = \sqrt{9} = 3$

$\hat{n} = \frac{1}{3}(1, 2, 2)$ , $v_{0n} = \frac{1}{3}(1+2) = 1$ m/s

$$\delta\rho_1 = -\frac{1{,}29}{(1-v_{\Gamma_1})}\,\delta v_n = -\frac{1{,}29}{1{,}245}\,\alpha \simeq 1{,}036\,\alpha$$

$\underline{\delta v'} = \frac{\alpha}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}$ , $\delta v' \cdot \hat{n} = \alpha$

$$\delta\rho_2 = -\frac{1{,}29}{-1{,}245}\,\alpha \simeq 1{,}036\,\alpha$$

(6.2) La deformazione di un corpo elastico, omogeneo ed isotropo è descritta dalle relazioni $u_1 = u_2 = 0$ , $u_3 = x_3\,\varphi(r)$ con $r = \sqrt{x_1^2 + x_2^2}$ e $\varphi \in C^2$. Il corpo è in equilibrio sotto l'azione di una forza di massa di componenti $F_\alpha = k x_\alpha$ , $\alpha = 1, 2$, $k$ costante. Determinare le funzioni $\varphi$ e $F_3$.

(32)

$$\Delta u_1 = 0 = \Delta u_2, \quad \boxed{\frac{\partial u_3}{\partial x_2} = \frac{\partial (x_3 \varphi(r))}{\partial x_2} = x_3 \varphi'(r) \frac{\partial r}{\partial x_2} =}$$

$$\boxed{= x_3 \varphi'(r) \frac{1}{\cancel{2}} \frac{\cancel{2} x_2}{r} = \frac{x_3 x_2}{r} \varphi'(r)}, \quad \boxed{\frac{\partial u_3}{\partial x_3} = \varphi(r),}$$

$$\frac{\partial^2 u_3}{\partial x_3^2} = 0, \quad \cancel{\frac{\partial^2 u_3}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(\frac{x_3 x_2}{r} \varphi'(r)\right)} \qquad \boxed{\varphi' = \frac{d\varphi}{dr}}$$

$$\frac{\partial^2 u_3}{\partial x_\beta \partial x_\alpha} = \frac{\partial}{\partial x_\beta}\left(\frac{\partial u_3}{\partial x_\alpha}\right) = \frac{\partial}{\partial x_\beta}\left(\frac{x_3 x_\alpha}{r} \varphi(r)\right) =$$

$$= x_3 \left[\delta_{\alpha\beta} \frac{\varphi'}{r} + x_\alpha \varphi'(r) \frac{\partial r^{-1}}{\partial x_\beta} + \frac{x_\alpha}{r} \frac{\partial \varphi'}{\partial x_\beta}\right) =$$

$$= x_3 \left[\delta_{\alpha\beta} \frac{\varphi'(r)}{r} - \frac{x_\alpha \varphi'(r)}{r^{2}} \frac{1}{\cancel{2}} \frac{\cancel{2} x_\beta}{r} + \frac{x_\alpha}{r} \varphi''(r) \frac{\partial r}{\partial x_\beta}\right] =$$

$$= x_3 \left[\delta_{\alpha\beta} \frac{\varphi'}{r} - \frac{x_\alpha x_\beta \varphi'}{r^3} + \frac{x_\alpha x_\beta}{r^2} \varphi''\right]$$

$$\frac{\partial^2 u_3}{\partial x_\alpha^2} = x_3 \left[\frac{2\varphi'}{r} - \frac{(x_1^2 + x_2^2)\varphi'}{r^3} + \frac{(x_1^2 + x_2^2)\varphi''}{r^2}\right] =$$

$$= x_3 \left[\frac{2\varphi'}{r} - \frac{\varphi'}{r} + \varphi''\right] = \boxed{\left(\varphi'' + \frac{\varphi'}{r}\right) x_3 = \Delta u_3}$$

$$\frac{\partial^2 u_3}{\partial x_\alpha \partial x_3} = \frac{\partial \varphi}{\partial x_\alpha} = \varphi'(r) \frac{\partial r}{\partial x_\alpha} = \frac{\varphi'(r)\, x_\alpha}{r}.$$

$$0 = \rho b_i + \mu \Delta u_i + (\nu+\mu)\ \text{grad div}\, u$$

$$\begin{cases} 0 = \rho b_1 + \mu \Delta u_1 + (\nu+\mu) \dfrac{\partial^2 u_j}{\partial x_1 \partial x_j} \\ 0 = \rho b_2 + \mu \Delta u_2 + (\nu+\mu) \dfrac{\partial^2 u_j}{\partial x_2 \partial x_j} \\ 0 = \rho b_3 + \mu \Delta u_3 + (\nu+\mu) \dfrac{\partial^2 u_j}{\partial x_3 \partial x_j} \end{cases}$$

$$\begin{cases} 0 = \rho H x_1 + (\nu+\mu) \dfrac{\partial^2 u_3}{\partial x_1 \partial x_3} = \rho H x_1 + (\nu+\mu) \dfrac{x_1 \varphi'(r)}{r} \\ 0 = \rho H x_2 + (\nu+\mu) \dfrac{\partial^2 u_3}{\partial x_2 \partial x_3} = \rho H x_2 + (\nu+\mu) \dfrac{x_2 \varphi'(r)}{r} \\ 0 = \rho b_3 + \left(\varphi'' + \dfrac{\varphi'}{r}\right) x_3 + 0 \end{cases}$$

$$\begin{cases} b_3 = -\dfrac{x_3}{\rho}\left(\varphi'' + \dfrac{\varphi'}{r}\right) = -\dfrac{x_3}{\cancel{\rho}}\left(-\dfrac{\rho H}{\nu+\mu} - \dfrac{\rho H}{(\nu+\mu)} \dfrac{\cancel{r}}{\cancel{r}}\right) = \boxed{\dfrac{2 H x_3}{\nu+\mu} = b_3} \\ x_1\left(\rho H + (\nu+\mu) \dfrac{\varphi(r)}{r}\right) = 0 \overset{x_1 \neq 0}{\iff} \dfrac{d\varphi}{dr} \dfrac{1}{r} = -\dfrac{\rho H}{(\nu+\mu)} \\ x_2\left(\rho H + (\nu+\mu) \dfrac{\varphi'(r)}{r}\right) = 0 \end{cases}$$

$$\Downarrow$$

$$d\varphi = -\frac{\rho H}{(\nu+\mu)}\, r\, dr$$

$$\Rightarrow \varphi(r) = -\frac{\rho H}{(\nu+\mu)} \frac{r^2}{2} + C \Rightarrow u_3 = x_3 \left[C - \frac{\rho H}{(\nu+\mu)} \frac{r^2}{2}\right]$$

$$\varphi' = -\frac{\rho H}{(\nu+\mu)} r, \quad \varphi'' = -\frac{\rho H}{\nu+\mu}$$

Es.6.6) Materiale elastico, omog., iso. in $C \equiv \{ \underline{x} \mid d_2^2 \le x_1^2+x_2^2+x_3^2 \le d_1^2 \}$

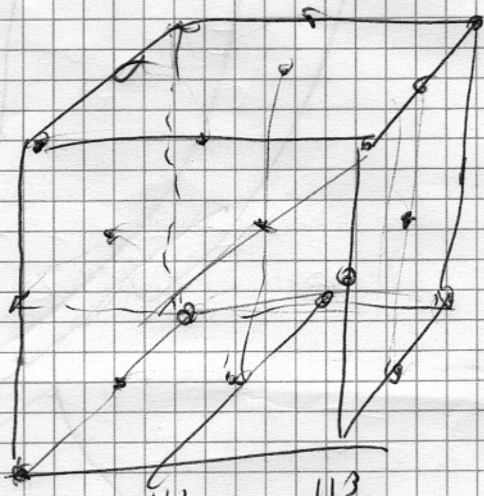
$\equiv \{ \underline{x} \mid |\underline{x}| = r ,\ d_2^2 < r^2 < d_1^2 \}$ con $d_1 > d_2$.

Il corpo C è soggetto a pressioni $p_2$ interne e $p_1$ esterne costanti. Studiare l'equilibrio del corpo in assenza di forze di massa. Cercare soluzioni della forma $\boxed{u_i = x_i \, \varphi(r)}$ con $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$.

Calcolare $\boxed{(\nu+\mu)\dfrac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu \dfrac{\partial^2 u_i}{\partial x_j \partial x_j} = 0}$ $\quad i = 1,2,3$

$i=1$: $(\nu+\mu)\dfrac{\partial^2 u_j}{\partial x_1 \partial x_j} + \mu \dfrac{\partial^2 u_1}{\partial x_j \partial x_j} = 0$

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j}\varphi(r) + x_i \frac{\partial \varphi(r)}{\partial x_j} = \delta_{ij}\varphi(r) + x_i \frac{x_j}{r}\varphi'(r)$$




$$\frac{\partial^2 u_i}{\partial x_k \partial x_j} = \delta_{ij}\frac{\partial \varphi}{\partial x_k} + \delta_{ik}\frac{x_j}{r}\varphi' + x_i \delta_{jk}\frac{\varphi'}{r} + \frac{x_i x_j}{r}\varphi''(r)\frac{x_k}{r} - \frac{x_i x_j \varphi'(r)}{r^2}\frac{x_k}{r} = \delta_{ij}\frac{x_k}{r}\varphi' + \delta_{ik}\frac{x_j}{r}\varphi' + \delta_{jk}\frac{x_i}{r}\varphi' + \frac{x_i x_j x_k}{r^2}\varphi'' - \frac{x_i x_j x_k}{r^3}\varphi'(r) = \frac{\varphi'}{r}\left(\delta_{ij}x_k + \delta_{ik}x_j + \delta_{jk}x_i - \frac{x_i x_j x_k}{r^2}\right) + \frac{x_i x_j x_k}{r^2}\varphi''$$

$i,j,k = 1,2,3$

$$\frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{\varphi'}{r}\left(\delta_{ij}x_j + \delta_{ij}x_j + \delta_{jj}x_i - \frac{x_i x_j x_j}{r^2}\right) +$$

$$+ \frac{x_i x_j x_j}{r^2}\varphi'' = \frac{\varphi'}{r}\left(x_i + x_i + 3x_i - x_i\frac{\cancel{r^2}}{\cancel{r^2}}\right) + \frac{x_i \cancel{r^2}}{\cancel{r^2}}\varphi'' =$$

$$= 4x_i\frac{\varphi'}{r} + x_i\varphi'' = \boxed{x_i\left(\varphi'' + 4\frac{\varphi'}{r}\right) = (\Delta u)_i}$$

$$\frac{\partial^2 u_j}{\partial x_k \partial x_j} = \frac{\varphi'}{r}\left(3x_k + x_k + x_k - \frac{\cancel{x_j x_j} x_k}{\cancel{r^2}}\right) + \varphi''\frac{\cancel{x_j x_j} x_k}{\cancel{r^2}}$$

$$= \boxed{x_k\left(\varphi'' + 4\frac{\varphi'}{r}\right) = \left[\mathrm{grad}(\mathrm{div}\, \underline{u})\right]_k}$$

Sostituendo abbiamo $\Theta_i = \mu\, x_i\left(\varphi'' + 4\frac{\varphi'}{r}\right) +$

$$+ (\nu+\mu)\, x_i\left(\varphi'' + 4\frac{\varphi'}{r}\right) = (\nu + 2\mu)\, x_i\left(\varphi'' + 4\frac{\varphi'}{r}\right)$$

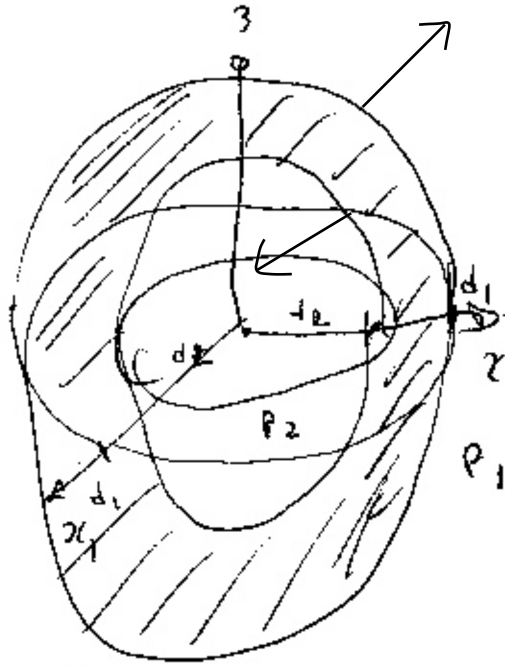
$$\psi = \varphi' \qquad \frac{d\psi}{dr} = -4\frac{\psi}{r} \iff \frac{d\psi}{\psi} = -4\frac{dr}{r}$$

$$\lg\psi = -4\lg r + \lg\psi_0 = \lg\left(r^{-4}\psi_0\right)$$

$$\psi = r^{-4}\psi_0, \quad \frac{d\varphi}{dr} = \psi_0 r^{-4}, \quad d\varphi = \psi_0 r^{-4} dr$$

$$\varphi - \varphi_0 = -\frac{1}{3}r^{-3}\psi_0 \implies \varphi(r) = A_1 r^{-3} + A_2$$

$$\implies u_i = x_i\left(A_1 r^{-3} + A_2\right)$$




$\underline{b=0}$ , $u_i = x_i\,\varphi(r)$

$$\boxed{-p_1 n_i = \nu \frac{\partial u_k}{\partial x_k} n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j}$$

per $r = d_1$

$\tau_2$ per $r = d_2$

$$-p_2 n_i = \nu \frac{\partial u_k}{\partial x_k} n_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j$$

$$\frac{\partial u_i}{\partial x_j} = \delta_{ij}\,\varphi(r) + \frac{x_i x_j}{r}\varphi'(r)$$

$r = d_1$ ,
$$\frac{\partial u_i}{\partial x_j}(d_1) = \delta_{ij}\,\varphi(d_1) + \frac{x_i x_j}{d_1}\varphi'(d_1) =$$

$$= \delta_{ij}\left(A_1 d_1^{-3} + A_2\right) - \frac{x_i x_j}{d_1} 3 A_1 d_1^{-4} =$$

$$= \delta_{ij}\left(A_2 + \frac{A_1}{d_1^3}\right) - 3\frac{x_i x_j A_1}{d_1^5}$$

$$(\operatorname{div} u) = \frac{\partial u_i}{\partial x_i}(d_1) = \delta_{ii}\left(A_2 + \frac{A_1}{d_1^3}\right) - \frac{3 x_i x_i A_1}{d_1^5}$$

$$= 3A_2 + \cancel{\frac{3A_1}{d_1^3}} - \cancel{3\frac{d_1^2}{d_1^5}A_1} = 3A_2$$

(1) su $d_1$
$$-p_1 n_i = \nu\, 3A_2 n_i + \mu\left[\delta_{ij}\left(A_2 + \frac{A_1}{d_1^3}\right) - 3\frac{x_i x_j A_1}{d_1^5} + \right.$$

$$\left. + \delta_{ji}\left(A_2 + \frac{A_1}{d_1^3}\right) - 3\frac{A_1 x_j x_i}{d_1^5}\right] n_j = 3\nu A_2 n_i + 2\mu n_i\left(A_2 + \frac{A_1}{d_1^3}\right) -$$

$$- 6\mu x_i x_j \frac{A_1}{d_1^5} n_j = \left[3\nu A_2 + 2\mu\left(A_2 + \frac{A_1}{d_1^3}\right)\right] n_i - 6\mu A_1 \frac{x_i}{d_1^5}(\underline{x}\cdot n)$$

$\varphi = r^2 - d_1^2 = x_1^2 + x_2^2 + x_3^2 - d_1^2 \qquad \nabla\varphi = (2x_1, 2x_2, 2x_3)$

normale esterna

$\hat{n} = |\nabla\varphi|^{-1}\nabla\varphi = \frac{\cancel{2}}{\cancel{2}r}(x_1, x_2, x_3)$ e sostituendo $\left(x \cdot n = \frac{x_1^2+x_2^2+x_3^2}{r} = r\right.$

$\hat{n}_1 = \hat{n}(d_1)$

$-p_1 \frac{x_i}{d_1} = \left[3\nu A_2 + 2\mu\left(A_2 + \frac{A_1}{d_1^3}\right)\right]\frac{x_i}{d_1} - 6\mu A_1 \frac{x_i}{d_1^4} \iff$ sulla superficie interna $\hat{n}^2 = -\hat{n}(d_1)$

$$\begin{cases} -p_1 = 3\nu A_2 + 2\mu\left(A_2 + \frac{A_1}{d_1^3}\right) - 6\frac{\mu A_1}{d_1^3} = A_2(3\nu+2\mu) - 4\mu A_1 d_1^{-3} \\ -p_2 = A_2(3\nu+2\mu) - 4\mu A_1 d_2^{-3} \end{cases}$$

$p_2 - p_1 = 4\mu A_1\left(d_2^{-3} - d_1^{-3}\right) \iff A_1 = \frac{p_2 - p_1}{4\mu\left(d_2^{-3} - d_1^{-3}\right)}$

$p_2 d_1^{-3} - p_1 d_2^{-3} = A_2(3\nu+2\mu)\left(d_2^{-3} - d_1^{-3}\right) \iff A_2 = \frac{p_2 d_1^{-3} - p_1 d_2^{-3}}{(3\nu+2\mu)\left(d_2^{-3} - d_1^{-3}\right)}$